S.No. 245 **17PMA01**

(For the candidates admitted from 2017–2018 onwards)

M.Sc. DEGREE EXAMINATION,
NOVEMBER-2020.

First Semester

Mathematics

LINEAR ALGEBRA

Time : Three hours Maximum : 75 marks

PART A — (10 × 2 = 20 marks)

Answer ALL the questions.

1. Define null space.

2. Prove that $L_\alpha(cf+g)=cL_\alpha(f)+L_\alpha(g)$.

3. Define algebraically closed field.

4. Define ring.

5. Define permutation of degree n.

6. Prove that similar matrices have the same characteristic polynomial.

7. When two matrices are said to be equivalent?

8. Define nilpotent linear operator.

9. Define T-annihilator of α.

10. State cyclic decomposition theorem.

PART B — ($5 \times 5 = 25$ marks)

Answer ALL the questions.

11. (a) Let T be a linear transformation from V into W. Then prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

Or

(b) Let V be a finite-dimensional vector space over the field F. For each vector α in V define $L_\alpha(f) = f(\alpha), f$ in V^*. Prove that the mapping $\alpha \to L_\alpha$ is an isomorphism of V onto V^{**}.

12. (a) Suppose f, g and h are polynomials over the field F such that $f \neq 0$ and $fg = fh$. Then prove that $g = h$.

Or

(b) Let D be a 2-linear function with the property that $D(A) = 0$ for all 2×2 matrices A over K having equal roots. Then prove that D is alternating.

13. (a) Let K be a commutative ring with identity, and let A and B be $n \times n$ matrices over K. Then prove that
$\det(AB) = (\det A)(\det B)$.

Or

(b) Let T be a linear operator on a finite-dimensional space V. Let $c_1, \ldots\ldots, c_k$ be the distinct characteristic values of T and let W_i be the null space of $(T - c_i I)$. Prove that the following are equivalent.

(i) T is diagonalizable.

(ii) The characteristic polynomial for T is

$$f = (x - c_1)^{d_1} \ldots (x - c_k)^{d_k} \text{ and}$$

$$\dim W_i = d_i,\ i = 1, \ldots., k$$

(iii) $\dim W_1 + \ldots + \dim W_k = \dim V$.

14. (a) Let W be an invariant subspace for T. The characteristic polynomial for the restriction operator T_W divides the characteristic polynomial for T. Prove that the minimal polynomial for T_W divides the minimal polynomial for T.

Or

(b) Define the following.

(i) Invariant subspace.

(ii) Projection of a vector space.

15. (a) If T is a nilpotent linear operator on a vector space of dimension n, then prove that the characteristic polynomial for T is x^n.

Or

(b) If A is the companion matrix of a monic polynomial p, then prove that p is both the minimal and characteristic polynomial of A.

PART C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. Let V be a finite dimensional vector space over the field F and let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \ldots \beta_n$ be any vectors in W. Then prove that there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j$, $j = 1, \ldots, n$.

17. If F is a field, prove that a non-scalar monic polynomial in $F[x]$ can be factored as a product of monic primes in $F[x]$ in one and only one way.

18. Let T be a linear operator on an n-dimensional vector space V. Prove that the characteristic and minimal polynomials for T have the same roots, except for multiplicities.

19. State and prove the primary decomposition theorem.

20. If M and N are equivalent $m \times n$ matrices with entries in $F[x]$, then prove that $\delta_k(M) = \delta_k(N)$, $1 \le k \le \min(m, n)$.
